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CONSTRUCTIONS OF MEASURES AND QUANTUM FIELD THEORIES ON MAPPING SPACES

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1. Introduction

My purpose here is to describe a simple construction of measures on spaces of maps from a topological space to a smooth manifold. As explained below, there are variants of the constructions in the case where the domain is a product with $\mathbb{R}$ that can be viewed as Euclidean path integrals for quantum field theories. (There are, of course, alternate constructions of measures on mapping spaces; for example, measures can be defined using results in [**6**], [**9**]. More recent examples can be found in [**13**], and in the work of Leandre [**7**, **8**].)

In what follows, M denotes the domain space. The sole input from M for the construction is a continuous function, $a : M \times M \to [0, \infty)$, which defines a non-negative definite kernel in the following sense: Fix any positive integer N; and choose any N distinct points $z_1, \ldots, z_N \in M$ and N real numbers $\{\eta_1, \ldots, \eta_N\}$. Then

$$\sum_{1 \le i \le j \le N} a(z_i, z_j)\eta_i\eta_j \ge 0. \tag{1.1}$$

Examples are given below.

Let X denote the range space, a smooth manifold with a given Riemannian metric. To simplify matters, X is assumed in what follows to be both compact and oriented. Use d to denote the dimension of X.

Suppose now that $\pi : P \to X$ is a compact, fiber bundle with the following additional data:

(1.2)

- A set, $\{\partial_1, \ldots, \partial_d\}$, of d vector fields that generate $TP/\text{kernel}(\pi_*)$ at each point.
- A volume form, dp, with total mass 1 and such that each vector field from this set has zero divergence.

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Note that the symbol dp is also used in what follows to indicate the product volume form on products of P.

Note that bundles such as P exist for any choice of X; for example, the principle $SO(d)$ bundle of oriented, orthonormal frames over X. Indeed, let Fr denote the latter. The metric's Levi–Civita connection endows TFr with the subbundle of horizontal frames, and since the points in Fr label the frames for TX, this bundle has a canonical trivialization. Moreover, the resulting vector fields are divergence free for the standard volume form.

The Construction: Some notation is required to set the stage. To start, suppose that a positive integer N has been choosen. In what follows, δ^N is used to denote the Dirac delta function on $\times_N P$ with support along the full diagonal. This is to say that $\delta^N\, dp$ is the measure on $\times_N P$ that sends a continuous function to

$$\int_{\times_N P} F\delta^N dp \equiv \int_p f(p,\dots,p)\, dp. \tag{1.3}$$

By the way, the notation here and below writes a measure as if it were a function; thus, the volume form is always present in the notation for integration.

To continue with the notation, for each $i \in \{1,\dots,N\}$ and $a \in \{1,\dots,d\}$, the symbol ∂_a^i denotes the vector field on $\times_N P$ that differentiates according to the basis vector ∂_a along the i'th factor of P. Suppose now that $z = (z_1,\dots,z_N) \in \times_N M$ is a chosen point. This point defines the differential operator

$$A_z \equiv \sum_{1\leq,j\leq N} a(z_i,z_j) \sum_{1\leq a\leq d} \partial_a^i \partial_a^j. \tag{1.4}$$

on $C^\infty(\times_N P)$.

By virtue of (1.1) and (1.2), this operator is negative semi-definite and symmetric. As a consequence, there exists a measure valued solution to the heat equation on $\times_N P$ that is characterized as follows: This solution, K_z, defines a continuous map from $[0,\infty)$ to the space of Borel measures on $\times_N P$ whose value at 0 is the measure δ^N. Moreover, K_z is such that when F is twice differentiable, then the pairing $\langle F, K_z\rangle$ is differentiable on $(0,\infty)$ where it obeys

$$\frac{d}{ds}\int_{\times_N P} FK_z|_s dp = \int_{\times_N P} (A_z F) K_z|_s\, dp. \tag{1.5}$$

Note that a theorem of Hormander [**3**] guarantees that K_z is smooth for $s > 0$ if the bilinear form in (1.1) is non-degenerate, and if the set of higher order Lie brackets of the vector fields in the set $\{\partial_a\}$ span TP at each point. In general,

(1.6)

- $K_z \geq 0$.
- $\int_P K_{(z_1,z_2,\ldots,z_N)}(p_1,\ldots,p_{N-1},p)dp = K_{(z_1,z_2,\ldots,z_{N-1})}(p_1,\ldots,p_{N-1})$.
- Let N and N' be positive integers with $N' \leq N$. If the final $N-N'+1$ entries of $z \in \times_N M$ are identical, then $K_z(p_1,\ldots,p_N) = \delta^{N-N'+1}(p_{N'},\ldots,p_N)K_{(z_1,\ldots,z_{N'-1},z_{N'})}(p_1,\ldots,p_{N'})$.
- Let σ simutaneously denote an element in the group of permutations of $\{1,\ldots,N\}$ and the action of this element on both $\times_N M$ and $\times_N P$. Then, $K_z = \sigma^*(K_{\sigma(z)})$.

Let P^M denote the space of all maps (continuous or not) from M to P. The collection of all such K_z can be used to define a measure on P^M as follows: The measure in question is defined on the σ-algebra that is generated by the 'cylinder' sets that are jointly labled by a positive integer, N, together with a collection of N pairs $\{(z_i,U_i)\}_{1\leq i\leq N}$ such that $z = (z_1,\ldots,z_N) \in \times_N M$ has distinct entries and each U_j is an open subset of P. The set labeled by the data $(N,\{(z_j,U_j)\})$ consists of the maps that send each z_j to its partnered set U_j. The measure of this set is deemed equal to

$$\int_{\times_{1\leq j\leq N}U_j} K_z|_{s=1}\, dp. \tag{1.7}$$

Granted the first, second and fourth points in (1.6), a theorem of Kolmogorov (see Theorem 1.10 in [**12**]) guarantees that the just asserted rules define a bonafide measure on P^M.

A push-forward measure is induced on X^M from the measure just described on P^M. To elaborate, this push-forward measure is defined by its values on certain cylinder sets of maps from M to X. Such a set is jointly labeled by a positive integer, N, together with a collection of N pairs $\{(z_j,V_j)\}_{1\leq j\leq N}$ where z is as above and where $V_j \subset X$ is an open set. The set with this label consist of those $\phi \in X^M$ such that $\phi(z_j) \in V_j$ for all $1 \leq j \leq N$. The measure of this set is given by the version of (1.7) where each U_j is taken to be the inverse image in P of the corresponding V_j.

It is of some interest to determine the support of the measures that are defined in this way. In this regard, the third point in (1.6) guarantees that the measure is supported on maps that are 'continuous' in a weak sense that is made precise in Lemma 2.3 to come. The measure is supported on the continuous maps M in various cases if the function a that appears in (1.1) is uniformly Holder continuous. For example, Theorem 2.4 asserts that the support is on the continuous functions when a is Holder continuous and M is a smooth, compact manifold.

By the way, the heat equations used here on the collection $\{\times_N P\}_{N=1,2,\ldots}$ are not the only ones that can be used to generate measures on P^M and X^M using a formula like that given in (1.7). The exploration of other versions are left for the reader.

A Gaussian measure: It is illuminating to compare the measure just defined with an average of push-forwards of the Gaussian measure on $\mathrm{Maps}\,(M;\mathbb{R}^d)$ with covariance function $C_{ab}(z,z') = a(z,z')\delta_{ab}$ with $\delta_{ab} = 1$ when $a = b$ and 0 otherwise. The expectations for this Gaussian are denoted by $(\,\cdot\,)$. Meanwhile, each $p \in P$ specifies a map, $\psi^p : \mathrm{Maps}\,(M;\mathbb{R}^d) \to \mathrm{Maps}\,(M;P)$ that is defined as follows: Let $z \to u(z) = (u^1(z),\ldots,u^d(z))$ denote a map from M to $\mathbb{R}^d$. Then $\psi^p(u)$ sends a given $z \in M$ to the time 1 point on the integral curve of the vector field $\Sigma_{1\leq a\leq d} u^a(z)\partial_a$ that starts at the point p. The 'Gaussian measure' on $\mathrm{Maps}\,(M;P)$ is the measure with expectations that are given by $\int_P (\psi^{p*}(\,\cdot\,))\,dp$. To make things explicit in the simpliest case, it is assumed in what follows that the inequality in (1.1) is strict unless all η_j are zero. Thus, the $N\times N$ matrix with $i{-}j$ component $a(z_i,z_j)$ is strictly positive definite in the case that the points $\{z_1,\ldots,z_N\}$ are pairwise distinct. This matrix is denoted hereby $\mathfrak{a}_z$, and $\mathfrak{a}_z^{-1}$ denotes its inverse.

When $u \in \mathbb{R}^d$, then $\sum_a u_a\partial_a$ defines a vector field on P. Let $T^u : P \to P$ denote the diffeomorphism that is obtained by integrating for time 1 this vector field. If f is a bounded function on P, then the assignment $(u,p) \to (T^{u*}f)(p)$ gives a bounded function on $\mathbb{R}^d \times P$. Of course, $T^{u*}f$ is smooth if f is.

Fix, N and $\{(z_j,U_j)\}_{1\leq j\leq N}$ as in the first construction. As explained above, such a collection labels a set of maps from M to P. The Gaussian measure of this set is given by the two equivalent expressions that follow:

(1.8)

- $$\int_P \left(\exp\left(\sum_{1\leq i,j\leq N} a(z_i,z_j) \sum_{1\leq a\leq d} \frac{\partial^2}{\partial u_a^i \partial u_a^j} \right) \cdot \prod_{1\leq j\leq N} (T^{u^j} * \chi_{U_j}) \right) \Bigg|_{u^1=\cdots=u^N=0} dp;$$
- $$\int_{\times_N \mathbb{R}^d} \left(\int_P \Pi_{1\leq j\leq N} (T^{u^j} * \chi_{U_j})\, dp \right)$$

$$\cdot \exp\left[-\frac{1}{4}\sum_{1\leq i\leq j\leq N}(\mathfrak{a}_z^{-1})_{i,j}\sum_{1\leq a\leq d}u_a^i u_a^j\right](4\pi)^{-N/2}\det(\mathfrak{a}_z)^{-1}\Pi_j du^j.$$

Here, χ_U denotes the characteristic function of a given set U. Meanwhile, du^j denotes the standard Lebesque measure on the j'th copy of $\mathbb{R}^d$. Granted that (1.8) defines a measure on P^M, there is then a corresponding push-forward measure on X^M.

The first measure and the Gaussian measure take the same input data to obtain a measure on P^M and X^M. In general, these measures are distinct. However, they do agree when the vector fields $\{\partial_a\}_{1\leq a\leq d}$ are pairwise commuting. Examples when this occurs arise when X has a flat metric. For these examples, the constructions given here can be used to produce measures that a physicist would recognize as a 'torroidal compactification' of a sort of free, bosonic sigma model (see, for example [**1**]).

An example of these construction arises in the case that $a(z,z')$ is positive and the Green's function for a self-adjoint, positive definite, elliptic operator on M. For example, in the case that $M = S^1$, the function a can be taken to equal a suitably positive constant plus the Green's function for the (positive definite) Laplacian. Note that the measure so defined on the space of loops in M is not, in general, Wiener measure.

In the case that $\dim(M) > 1$, the Green's function for the Laplacian is unbounded, and so the latter cannot be used without some sort of 'renormalization'. As it turns out, a renormalization prescription does exist in certain cases, for example when $X = G/H$ where G is a compact Lie group and $H \subset G$ is a subgroup. The renormalization issue will be discussed in a planned sequel to this article.

The case that $M = \mathbb{R}\times Y$ is rather special by virtue of the fact that measures with a property called reflection positivity provide a quantum field theory with Hamiltonian. As is explained in Section 4, examples of measures from the constructions given here provide reflection positive measures and thus quantum field theories. In particular, this is the case when Y is a smooth, compact manifold, and when $a(z,z')$ is a positive Green's function for an operator on $\mathbb{R}\times Y$ of the form

$$\frac{d^2}{dt^2} + \mathfrak{L} \tag{1.9}$$

with $\mathfrak{L}$ a negative definite, self-adjoint elliptic differential operator on Y whose order is greater than twice the dimension of Y. The fact that the Gaussian measure provides a reflection positive quantum field theory is well known (see e.g., Chapter 6 of [**2**]).

What follows briefly explains how a physicist might think of the measures that are defined by either of these constructions. In this regard, both measures can be viewed as defining a Euclidean version of a 'non-linear σ model'. To elaborate, the fields are the maps from M to X, and the measure is motivated by imagining a 'Euclidean Feynman path integral' that is defined by the Lagrangian that arises as follows: A map from M to X is given by specifying a point $p \in P$ together with a map, $\phi : M \to \mathbb{R}^d$ and then pushing forward the map from M to P that sends $z \in M$ to $T^{\phi(z)}(p)$. A Lagrangian for such maps is defined by the bilinear form that sends ϕ to the pairing $\phi \to \langle \phi, \mathcal{D}\phi \rangle_M$, where $\mathcal{D}$ is the operator inverse to that defined by using the function a as a bilinear form on $L^2(M) \times \mathbb{R}^d$. (One might have to pretend that such an inverse exists.) The constructions given above make rigorous a notion of integration on the maps from M to X for the 'volume' form

$$\exp\left(-\int_M \phi\mathcal{D}\phi\right) d^\infty\phi\, dx. \tag{1.10}$$

Quantum field theories that are, in a formal sense, much like those described here for the case that $Y = S^1$ and $\mathfrak{L}$ is second order are realized by certain sorts of conformal field theories. See, for example the lectures of Gawedzki [**1**].

The remainder of this article is organized as follows: The next section provides the details for the first construction of a measure on P^M. The subsequent section discusses various corresponding points for the Gaussian measure. The fourth section discusses the relation to quantum field theories in the case that $M = \mathbb{R} \times Y$. The fifth section contains the proof of the main theorem in Section 4. The final section describes certain generalizations of the first construction.

There may be sequels to this article that discuss the renormalization question, 'supersymmetric' versions of the constructions, and the use of these measures to make a Hilbert space context for a Dirac operator on the space of maps from S^1 to X.

This introduction ends by acknowledging debts to Curt McMullen and to Dan Stroock for sharing their considerable insight on various analytic issues. Dan Stroock is also thanked for his comments on an early version of the manuscript.

2. The details of the construction

The first order of business is to establish certain properties of K_z. The lemma that follows contains the basic existence and uniqueness statement. To set the stage, suppose that a positive integer N is fixed along with the function in (1.1).

Lemma 2.1. *There is a unique measure valued solution to (1.5) in all cases. This is to say that there there exists a unique, continuous map, $s \to \int_{\times_N P}(\cdot)K_z dp$, from $[0,\infty)$ into the space of Borel measures on P which obeys*

$$\frac{d}{ds}\int_{\times_N P} FK_z\, dp = \int_{\times_N P} A_z FK_z\, dp$$

with

$$\int_{\times_N P} FK_z|_{a=0}\, dp = \int_{\times_N P} F\delta^N\, dp.$$

Proof of Lemma 2.1. Arguments for existence and uniqueness can be taken almost verbatim from the proof of Theorem 3.2.6 in the book by Stroock and Varadhan [**12**]. q.e.d.

As remarked, Hormander [**3**] guarantees that K_z defines a smooth function on $(0,\infty)\times(\times_N P)$ when two conditions hold: First, (1.1) defines a non-degenerate bilinear form. Second, the set

$$\{\partial_a, [\partial_a,\partial_b], [\partial_a,[\partial_b,\partial_c]], \dots\} \tag{2.1}$$

of finite commutators spans TP at each point of P. Note that this second condition is guaranteed in the case that P is the frame bundle of X and the metric on X is chosen in a suitably generic fashion. It is also guaranteed in the case that P is a compact, simple Lie group and X is the quotient of P by some compact subgroup.

The positivity of K_z as claimed in the first point of (1.6) follows from a version of the maximum principle that is discussed in Chapter 3.1 of [**12**]. The remaining points in (1.6) follow as corollaries to the uniqueness assertion in Lemma 2.1.

The next lemma addresses the behavior of K_z as z is varied in $\times_N M$.

Lemma 2.2. *Fix a postive integer N and a continuous function F on $\times_N P$. Then, the assignment of $\int_{\times_N P} FK_z|_s dp$ to a given $(s,z)\in[0,\infty)\times(\times_N M)$ defines a continuous function on $[0,\infty)\times(\times_N M)$.*

Proof of Lemma 2.2. The assertion is proved with arguments that are essentially the same as those used in [**12**] to prove the latter's Theorem 3.2.6. q.e.d.

As noted in the introduction, the first, second and fourth points in (1.6) guarantee via a theorem of Kolmogorov that the collection of all $K_z|_{s=1}$ where z has pairwise distinct entries define a probability measure, $\mathcal{P}$, on P^M. Integration with respect to this measure is denoted in what follows by $\langle\cdot\rangle$. Functions of the following sort are integrable with respect to this measure: Fix a positive integer N, a point $z \in \times_N M$ and a continuous function F on $\times_N P$; then, define $F_z : P^M \to \mathbb{R}$ via the rule

$$F_z(\phi) = F(\phi(z_1)\ldots,\phi(z_N)). \tag{2.2}$$

The integral of F_z with respect to the probability measure $\mathcal{P}$ is denoted by $\langle F_z\rangle$ and it is defined to equal

$$\langle F_z\rangle \equiv \int_{\times_N P} FK_z|_{s=1}dp \tag{2.3}$$

in the case that z has pairwise distinct entries. If two or more entries of z agree, then Kolmogorov's construction defines the integral of F_z via an appropriate $N' < N$ version of $K_{(\cdot)}$. Even so, Lemma 2.2 together with the third and fourth points in (1.6) guarantee that the integral of F_z varies continuously with variations of z in $\times_N M$ including those that cross a diagonal.

With the preceding understood, what follows speaks to the continuity of the maps in the support of the just defined measure on P^M.

Lemma 2.3. *Fix a positive integer N, a function F on $\times_N P$, and positive numbers ε and δ. There is a neighborhood of the diagonal embedding of $\times_N M$ in $(\times_N M)\times(\times_N M)$ whose points have the following property: If (z,w) is in this neighborhood, then the set of maps in P^M where $|F_z(\cdot)-F_w(\cdot)|>\delta$ has measure less than ε.*

Proof of Lemma 2.3. The measure of the set of maps where $|F_z(\cdot) - F_w(\cdot)| > \delta$ is no greater than

$$\begin{aligned}\delta^{-2}\langle F_zF_z - 2F_zF_w + F_wF_w\rangle = \delta^{-2}&\left\langle (F\otimes F)_{(z,z)} - (F\otimes F)_{(z,w)}\right\rangle \\ &+ \left\langle (F\otimes F)_{(w,w)} - (F\otimes F)_{(w,z)}\right\rangle,\end{aligned} \tag{2.4}$$

where the notation is such that $F\otimes F$ is the function on $\times_{2N}P \equiv (\times_N P)\times(\times_N P)$ that assigns $F(z)F(w)$ to any given pair (z,w). Now, by virtue of Lemma 2.3 and the last two points in (1.6), the assignment of $(z,w)\in(\times_N M)\times(\times_N M)$ to $\langle (F\otimes F)_{(z,z)} - (F\otimes F)_{(z,w)}\rangle$ defines a continuous function. As it vanishes on the diagonal embedding of $(\times_N M)$, it must have small absolute value on some neighborhood of the diagonal. q.e.d.

The measure defined here has support on the continuous maps from M to P if the function a on $M\times M$ is suitably regular. To give some sense of what is required, suppose in what follows that M is a smooth, compact manifold.

Theorem 2.4. *The measure on P^M constructed above induces a measure on the space of continuous maps from M to P if the function a that appears in* (1.1) *is Holder continuous for some positive exponent.*

Proof of Theorem 2.4. The proof of this proposition uses much the same strategy that is used to prove Theorem 2.1.6 in [**12**]. Moreover, all arguments that follow have antecedents in the latter proof.

The proof is broken into seven parts.

Part 1: This part supplies an analog of an observation of Kolmogorov that appears as Lemma 2.1.2 in [**12**].

Lemma 2.5. *A probability measure on P^M induces a probability measure on the subset in P^M of continuous maps from M to P if the following is true: Let $S \subset M$ denote any countable set. Then, the measure assigns probability 1 to the set of maps that are uniformly continuous on S.*

Proof of Lemma 2.5. According to Lemma 2.1.1 in [**12**], the measure induces a probability measure on a given subset of P^M if the subset in question has outer measure 1. This means that every set in the σ-algebra that contains the given subset has measure 1. Meanwhile, a set in the σ-algebra is determined by the values of the maps on some countable subset $S \subset M$. This understood, let $\mathcal{O}'$ denote a set in the σ-algebra that contains the continuous maps, and let S' denote a corresponding countable subset that defines $\mathcal{O}'$. Let $S \subset M$ denote a countable, dense set that contains S'; and let $\mathcal{O} \subset \mathcal{O}'$ denote the subset in $\mathcal{O}'$ of maps ϕ such that $\phi|_S$ is uniformly continuous. Then $\mathcal{O}$ contains the continuous maps and so it is enough to prove that $\mathcal{O}$ has measure 1. To see that there are no additional, independent constraints for membership in $\mathcal{O}$, let $\phi : S \to P$ denote any given, uniformly continuous map. As ϕ is uniformly continuous, it extends as a uniformly continuous map to the whole of M since S is dense in M. q.e.d.

Part 2: This part of the argument for Theorem 2.4 supplies a multi-dimensional generalization of Theorem 2.1.3 in [**12**].

Proposition 2.6. *Suppose that C is a smooth, compact Riemannian manifold with boundary, and set m to denote the dimension of C. Suppose, in addition, that $\alpha \in (0,1)$ and $n \geq 2$ are such that $n\alpha > 2.1m$. There is a constant, $\kappa = \kappa(C, n, \alpha)$, with the following significance: Let θ denote a Lipschitz function with compact support in C. Set*

$$Q(\theta) \equiv \iint_{C\times C} \frac{|\theta(w) - \theta(v)|^{2n}}{\operatorname{dist}(w,v)^{2n\alpha}}\, dw\, dv. \tag{2.5}$$

Then, $|\theta(x) - \theta(y)| \le \kappa Q(\theta)^{1/2n} \cdot \operatorname{dist}(x, y)^{\alpha - m/n}$ *for all pairs* $(x, y) \in C \times C$.

Remark that the lower bound $n\alpha \ge 2.1m$ is almost surely not optimal. It may be that a version of this proposition holds as long as $n\alpha > m$.

Proof of Proposition 2.6. The $m = 1$ assertion is a special case of Theorem 2.1.3 of [**12**]. What follows is an argument that works for any m. To start, take note that it is sufficient to consider the case where C is the unit cube in $\mathbb{R}^m$ with its flat metric. Indeed, the general case can be reduced to the latter by employing a partition of unity whose constituent functions have support in coordinate charts. This understood, let C now denote the unit cube in $\mathbb{R}^m$ and let θ denote a function on this cube with compact support.

Assume that $x \ne y$ are points in C and set W to denote the ball of radius $\frac{1}{4}|x - y|$ centered on y. Define

$$B(x|y) \equiv \int_W \frac{|\theta(x) - \theta(w)|^n}{|x - w|^{n\alpha}}\, dw. \tag{2.6}$$

Let $B(x) \equiv \sup_{y\in C} B(x|y)$ and let $B = \sup_{x\in C} B(x)$. Subsequent arguments obtain a bound on B of the form $B \le \upsilon Q(\theta)^{1/2}$ with υ being independent of θ, and the latter bound is then used to obtain the desired Holder bound for the function θ.

To obtain the Holder bound for θ from a bound on B, fix $x \ne y$ and set $d = |x-y|$. Introduce $V \equiv V(y)$ to denote the ball of radius $\frac{d}{8}$ whose center is the halfway point of the line segment between x and y. Then

$$|\theta(x)-\theta(y)| \le \left(\frac{8}{d}\right)^m \frac{1}{\mu}\int_V |\theta(x)-\theta(v)|dv + \left(\frac{8}{d}\right)^m \frac{1}{\mu}\int_V |\theta(y)-\theta(v)|dv, \tag{2.7}$$

where μ here denotes the volume of the unit ball in $\mathbb{R}^m$. Because the distance between any given $z \in V$ and x is at most $\frac{5d}{8}$, the inequality in (2.7) implies that

$$\begin{aligned} |\theta(x) - \theta(y)| \le{}& \left(\frac{8}{d}\right)^m \left(\frac{5d}{8}\right)^\alpha \frac{1}{\mu}\int_V \frac{|\theta(x) - \theta(v)|}{|x - v|^\alpha} dv \\ &+ \left(\frac{8^m}{d}\right) \left(\frac{5d}{8}\right)^\alpha \frac{1}{\mu}\int_V \frac{|\theta(y) - \theta(v)|}{|y - v|^\alpha} dv. \end{aligned} \tag{2.8}$$

An application of Holder's inequality to the right-hand side of (2.8) yields

$$|\theta(x) - \theta(y)| \le \left(\frac{8}{d}\right)^{m/n} \left(\frac{5d}{8}\right)^\alpha \left(\frac{1}{\mu}\right)^{1/n} \Bigg[\left(\int_V \frac{|\theta(x) - \theta(v)|^n}{|x - v|^{n\alpha}} dv\right)^{1/n} \tag{2.9}$$

$$+ \left(\int_V \frac{|\theta(y) - \theta(v)|^n}{|y - v|^{n\alpha}} dv \right)^{1/n} \Bigg].$$

Since neither term in the brackets on the left-hand side of (2.9) is greater than $B^{1/n}$, the latter expression gives the desired Holder bound for θ if it is the case that $B \le \upsilon Q(\theta)^{1/2}$.

Equation (2.9) is also the starting point for a derivation of bound, $B \le \upsilon Q(\theta)^{1/2}$. In particular, the first step towards bounding B is an application to (2.9) of the following observation: Given $\varepsilon > 0$, there exists a constant, $c(n, \varepsilon)$, such that

$$|q + r|^n \le (1 + \varepsilon) q^n + c(n, \varepsilon) r^n \text{ for any given } q, r \ge 0. \tag{2.10}$$

The application of (2.10) to (2.9) yields

$$\frac{|\theta(x) - \theta(y)|^n}{|x - y|^{n\alpha}} \le \left(\frac{8}{d}\right)^m \left(\frac{5}{8}\right)^{n\alpha} \frac{1}{\mu} (1 + \varepsilon) B(x) + \left(\frac{8}{d}\right)^m \left(\frac{5}{8}\right)^{n\alpha} \frac{1}{\mu} c(n, \varepsilon) \int_v \frac{|\theta(y) - \theta(v)|^n}{|y - v|^{n\alpha}} dv. \tag{2.11}$$

To proceed from here, fix a point $y_0 \ne x$ and let W now denote the ball of radius $\frac{1}{4}|x - y_0|$ with center at y_0. Integrate both sides of (2.11) with respect to the variable y with W being the domain of integration. The resulting integration yields the inequality

(2.12)

$$B(x|y_0) \le \left(\frac{8}{3}\right)^m \left(\frac{5}{8}\right)^{n\alpha} (1 + \varepsilon) B(x) + \left(\frac{5}{8}\right)^{n\alpha} \frac{1}{\mu} c(n, \varepsilon) \int_W \frac{1}{|x - w|^m} \left(\int_{V(w)} \frac{|\theta(y) - \theta(v)|^n}{|y - v|^{n\alpha}} dv \right) dw.$$

With a suitable choice for ε, this last inequality gives a bound for $B(x)$ in the case that

$$n\alpha > m \frac{\ln(8/3)}{\ln(8/5)}. \tag{2.13}$$

Indeed, an application of Holder's inequality to the right most double integral in (2.12) bounds the latter by a θ, x and y_0 independent multiple of $c(n, \varepsilon) \cdot Q(\theta)^{1/2}$. With (2.13) understood, an appropriately small choice for ε parlays the latter bound into an upper bound for $B(x)$ by a θ and x independent multiple of $Q(\theta)^{1/2}$. Of course, such an x-independent bound for $B(x)$ implies the desired bound for B. q.e.d.

Part 3: Fix an isometric embedding of P into some large k version of $\mathbb{R}^k$ and so identify P with a subset of $\mathbb{R}^k$. Likewise, pick an isometric embedding of M into some large dimensional Euclidean space. Let $\mathbb{R}^{k'}$ denote the latter.

Let $S \subset M$ denote a countable set. For each $N < \infty$, let S_N denote the subset indexed by the integers $\{1, \ldots, N\}$. Given $\phi \in P^M$, Section 2.2 of Chapter VI in [**11**] provides a continuous extension of ϕ's restriction to S_N as a map from the whole of M to $\mathbb{R}^k$. The extension, $\theta \equiv \theta_N[\phi]$, is given by a version of Equation (8) on page 172 of [**11**]:

$$\theta(z) = \sum_i \phi(z_i)\varphi_i^*(z) \text{ if } z \notin S_N \text{ and } \theta(z) = \phi(z) \text{ if } z \in S_N. \tag{2.14}$$

To explain, the sum for θ is indexed by the positive integers, with each integer labeling a k'-dimensional cube in $\mathbb{R}^{k'}$ that lies in the complement of S_N. Note that the diameter of each cube is greater than its distance to S_N but less than four times this distance. Meanwhile, interiors of distinct cubes are disjoint, and there is a constant, J, that is independent of N and S and is such that any given cube intersects at most J others. The collection $\{\varphi_i^*\}$ label a partition of unity for $\mathbb{R}^{k'} - S_N$ such that the support of φ_i^* lies in the union of i'th cube and the $J - 1$ others that intersect it. In particular, $\varphi_i^* = \varphi_i/(\sum_j \varphi_j)$ where φ_i is a translated and rescaled version of a standard compactly support function with value 1 on the unit cube. In the sum for $\theta(z)$, a given z_i is a point in S_N whose distance to z gives the distance from the i'th cube to S_N.

As explained in the aforementioned section of [**11**], the function θ is smooth on the complement of S_N and Lipschitz on the whole of M.

Part 4: Fix $\alpha > 0$ and n to satisfy the conditions of Proposition 2.6 with m set to equal the dimension of M. Now, introduce

$$\tau_N[\phi] \equiv \sup_{s \neq s' \in S_N} \frac{\mathrm{disp}_p(\phi(s), \phi(s'))}{\mathrm{disp}_M(s, s')^{\alpha - m/n}}. \tag{2.15}$$

Given $R > 0$, let $\mathcal{A}_{N,R} \subset P^M$ denote the subset of maps ϕ with the property that the value of the $C = M$ version of $Q(\cdot)$ in (2.5) on $\theta_N[\phi]$ is larger than R. Meanwhile, let $\mathcal{B}_{N,R}$ denote the subset of maps $\phi \in P^M$ where $\tau_N[\phi] \geq R$. An appeal to Proposition 2.6 finds a constant, γ, that is independent of N and R, and is such that the measure of $\mathcal{A}_{N,R}$ is greater than or equal to that of $\mathcal{B}_{N,R}$ for the case that $R' = \gamma R^{1/n}$.

The plan now is to prove the following:

Lemma 2.7. *If the function a that appears in* (1.1) *is Holder continuous on $M \times M$ with some positive exponent, then the following is true: Given $\varepsilon > 0$, there exists R_ε such that $\mathcal{A}_{N,R}$ has measure less than ε for all N and $R \geq R_\varepsilon$.*

Granted this lemma, here is how to complete the proof of Theorem 2.4: Let $\mathcal{B}_R$ denote the subset of maps $\phi \in P^M$ where $\tau_N[\phi] \geq R$ for all N. Thus, $\mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots$ and $\cup_N \mathcal{B}_{N,R} = \mathcal{B}_R$. This understood, it follows from Lemma 2.7 that the $R' = \gamma R^{1/n}$ version of $\mathcal{B}_R$ has measure bounded by ε. Thus, $\lim_{R\to\infty}$ measure $(\mathcal{B}_R) = 0$ and so $\phi \in P^M$ has probability 1 of being uniformly continuous on S. Now, invoke Lemma 2.5.

Part 5: This Part 5 together with Parts 6 and 7 contain the

Proof of Lemma 2.7. There are three steps to the proof. To start the first step, view θ in (2.14) as a random variable for the probability measure. As such, there is an R and N independent constant, c_1, such that

$$\text{measure}\,(\mathcal{A}_{N,R}) \leq c_1 R^{-1} \left\langle \iint_{M\times M} \frac{|\theta(w)-\theta(v)|^{2n}}{\operatorname{dist}(w,v)^{2n\alpha}} dv\, dw \right\rangle. \tag{2.16}$$

Integration over $M \times M$ can be done after integration on P^M without changing the right-hand side of (2.16); thus

$$\text{measure}\,(\mathcal{A}_{N,R}) \leq c_1 R^{-1} \iint_{M\times M} \frac{1}{\operatorname{dist}(w,v)^{2n\alpha}} \left\langle |\theta(v)-\theta(w)|^{2n} \right\rangle dv\, dw. \tag{2.17}$$

The task at hand is to find a useful upper bound on $\langle|\theta(v)-\theta(w)|^{2n}\rangle$. As is explained momentarily, if the function a that appears in (1.1) is Holder continuous with some exponent $\nu \in (0,1)$, then

$$\langle|\theta(v)-\theta(w)|^{2n}\rangle \leq c\, \operatorname{dist}(v,w)^{n,\nu}, \tag{2.18}$$

where c is independent of v and w. Given that (2.18) holds, take the constants n and α that appear in (2.16) so that $\alpha < \frac{1}{2}(\nu + m/n)$. With this choice, use (2.18) to obtain an R and N independent upper bound for the double integral that appears in (2.17). Such a bound finds the measure of $\mathcal{A}_{N,R}$ less than $c_2 R^{-1}$ with c_2 independent of R and N; and so, Lemma 2.7 follows. q.e.d.

Part 6: The bound in (2.18) is obtained with the help of the following lemma:

Lemma 2.8. *Suppose that $n \geq 2$. There is a constant, κ, with the following significance: If z and z' are distinct points in M, then*

$$\langle|\phi(z)-\phi(z')|^{2n}\rangle \leq \kappa(|(a(z,z)-a(z,z')|^n + |a(z',z')-a(z,z')|^n). \tag{2.19}$$

The proof of this lemma appears in Part 7 below. Accept it as true in this Part 6.

The derivation of (2.18) from (2.19) differs in the following two cases:

$$\text{(2.20)} \qquad \bullet \quad \operatorname{dist}(w,v) > \frac{1}{100}(\operatorname{dist}(v,S_N) + dist(w,S_N)).$$
$$\bullet \quad \operatorname{dist}(w,v) \le \frac{1}{100}(\operatorname{dist}(v,S_N) + \operatorname{dist}(w,S_N)).$$

To derive (2.18) when the first point in (2.20) is valid, start with the inequality

(2.21)
$$|\theta(v)-\theta(w)| \le |\phi(z_v)-\phi(z_w)| + \sum_j{}^{v} |\phi(z_j)-\phi(z_v)| + \sum_j{}^{w} |\phi(z_j)-\phi(z_w)|,$$

where

(2.22)

- z_v and z_w are in S_N with $\operatorname{dist}(z_v, z_w) \le c_2 \operatorname{dist}(v,w)$.
- The sum with superscript v has $J-1$ terms; and each z_j that appears is a point in S_N with $\operatorname{dist}(z_j, z_v) \le c_2 \operatorname{dist}(v,w)$.
- The sum with superscript w has $J-1$ terms, and each z_j that appears is a point in S_N with $\operatorname{dist}(z_j, z_w) \le c_2 \operatorname{dist}(v,w)$.

Note that the constant c_2 that appears here is independent of N, S, v and w. To obtain (2.22), let i and i' denote indices such that v is in cube i and w in the cube i'. Take $z_v = z_i$ and $z_w = z_i'$. Now, write

$$\text{(2.23)} \qquad \theta(v) = \phi(z_i) + \sum_j \varphi_j^*(v)(\phi(z_j) - \phi(z_i)),$$

and write a similar formula for $\theta(w)$. Use these to bound $|\theta(v) - \theta(w)|$ by

$$\text{(2.24)} \quad |\phi(z_i)-\phi(z_{i'})| + \sum_j \varphi_j^*(v)|\phi(z_j)-\phi(z_i)| + \sum_j \varphi^*(w)|\phi(z_j)-\phi(z_{i'}).$$

An inequality such as that given by (2.21) follows from (2.24) by setting $\varphi_j^*(\cdot)$ to equal 1 when it is not equal to zero.

To see how (2.22) comes about, note that the first point in (2.20) implies that $\operatorname{dist}(v,w)$ is greater than a uniform multiple of the diameter of both cube i and cube i'. As a consequence, a uniform multiple of $\operatorname{dist}(v,w)$ bounds $\operatorname{dist}(v,z_i)$ and likewise $\operatorname{dist}(w,z_{i'})$. Thus, a uniform multiple of $\operatorname{dist}(v,w)$ bounds $\operatorname{dist}(v,z_i) + \operatorname{dist}(w,z_{i'}) + \operatorname{dist}(v,w)$, and the latter bounds $\operatorname{dist}(z_i,z_{i'})$. This establishes the first point in (2.22). The second point follows by virtue of the fact that $\varphi_j^*(v) \ne 0$ only if the j'th and i'th cubes are adjacent; and if adjacent, then $\operatorname{dist}(z_i,z_j)$ is bounded by a uniform multiple of the diameter of the i'th cube. A similar argument gives the third point in (2.22).

Here is a direct consequence of (2.21):

(2.25)
$$\langle|\theta(v)-\theta(w)|^{2n}\rangle \le c_3(\langle|\phi(z_v)-\phi(z_w)|^{2n}\rangle + \sum_j{}^{v}\langle|\phi(z_j)-\phi(z_v)|^{2n}\rangle + \sum_j{}^{w}\langle|\phi(z_j)-\phi(z_w)|^{2n}\rangle),$$

where c_3 is again independent of N, S and both v and w. Granted (2.25), invoke (2.19) and then the assumption that a is Holder continuous with exponent ν leads directly to the desired (2.18).

Consider now the case that the second point in (2.20) is the relevant one. Let d denote the distance between v and S_N. As a consequence of the definitions in [**11**], the distance between w and S_N can be written as $c \cdot d$ where c and $1/c$ are uniformly bounded away from zero. This implies that all points z_k where either $\varphi_k^*(v)$ or $\varphi_k^*(w)$ is non-zero lie in a ball whose radius is bounded uniformly by d. Moreover,

$$\sup_z |d\varphi_k^*| \le c_5 d^{-1} \tag{2.26}$$

for all such k, where c_5 is independent of N, S, v and w.

With the preceding understood, let z' denote one such z_k and write

$$\theta(v)-\theta(w) = \sum_k{}'(\varphi_k^*(v)-\varphi_k^*(w))(\phi(z_k)-\phi(z'), \tag{2.27}$$

where the prime on the summation indicates that at most $2J$ non-zero terms are present. With (2.27) in hand, invoke (2.26) to conclude that

$$|\theta(v)-\theta(w)|^{2n} \le c_6 d^{-2n}\mathrm{dist}(v,w)^{2n}\sum_k{}'|\phi(z_k)-\phi(z')|^{2n}, \tag{2.28}$$

where c_6 is independent of N, S, v and w.

To proceed from here, use (2.28) with (2.19) to bound $\langle|\theta(v)-\theta(w)|^{2n}\rangle$ by

(2.29)
$$c_7 d^{-2n}\mathrm{dist}(v,w)^{2n}\sum_k{}'(|a(z_k,z_k)-a(z_k,z')|^n + |a(z',z')-a(z',z_k)|^n).$$

If the function a is Holder continuous with exponent $\nu \in (0,1)$, then (2.29) implies (2.18) when the second point in (2.20) holds. Indeed, this is so because both $\mathrm{dist}(v,w)$ and $\mathrm{dist}(z_k,z')$ are bounded in the case of the second point in (2.20) by a multiple of d that is independent of v, w, N and S.

Part 7: This part of the proof contains the promised

Proof of Lemma 2.8. The bound in (2.19) is proved with arguments that are much like those in [**12**] that prove Equation (1.10) in [**12**]'s Chapter 3.1. To start, fix a Riemannian metric on P that makes the vectors $\{\partial_a\}$ orthonormal. Let $\mathrm{dist}_P(\cdot,\cdot)$ denote the corresponding distance function on $P \times P$. Keep in mind that this function is smooth on some tubular neighborhood of the diagonal. Now, fix some smooth, non-negative function, F, on $P \times P$ with the following properties: First, F restricts to some small radius tubular neighborhood of the diagonal as $\mathrm{dist}_P(\cdot,\cdot)^2$. Second, F vanishes only on the diagonal. In particular, there should exist $\delta > 0$ such that $F \geq \delta$ on the complement inside $P \times P$ of some smaller radius tubular neighborhood of the diagonal.

Granted the preceding, note that Lemma 2.8 follows with a proof that

$$\langle F(\phi(z), \phi(z'))^n\rangle \leq \kappa'(|(a(z,z) - a(z,z')|^n + |a(z',z) - a(z,z')|^n), \tag{2.30}$$

with κ' a constant that is independent of z and z'. Keep in mind here and in what follows that the left-hand side of (2.30) is defined to be

$$\iint_{P\times P} F(p_1,p_2)^n K_{(z,z')}(p_1,p_2)|_{s=1}\, dp_1\, dp_2. \tag{2.31}$$

To obtain (2.30), set

$$\mathfrak{a} \equiv |(a(z,z) - a(z,z')|^n + |a(z',z') - a(z,z')|^n, \tag{2.32}$$

let $\mathfrak{b}$ denote a positive constant, and introduce the function G on $[0,\infty)\times P \times P$ given by

$$\mathrm{G}(s,p_1,p_2) = \mathfrak{a}(1-s)e^{\mathfrak{b}(1-s)} + F(p_1,p_2)^n e^{\mathfrak{b}(1-s)}. \tag{2.33}$$

Introduce as shorthand $A \equiv a(z,z)\partial_a^1\partial_a^1 + 2a(z,z')\partial_a^2\partial_a^1 + a(z',z')\partial_a^2\partial_a^2$. Here and below, the repeated subscript is implicitly summed over the range $1 \leq a \leq d$. As is argued momentarily, there is some (z,z')-independent choice for $\mathfrak{b}$ that guarantees the inequality

$$\partial_s\mathrm{G} + A\mathrm{G} \leq 0 \tag{2.34}$$

at all $s \in [0,1]$ and $(p,p') \in M$. By virtue of the fact that $K_{(z,z')} \geq 0$, the inequality in (2.34) implies that

$$\begin{aligned}\iint_{P\times P} & F(p_1,p_2)^n K_{(z,z')}(p_1,p_2)|_{s=1}\, dp_1\, dp_2 \\ &= \iint_{P\times P} \mathrm{G}K_{(Z,Z')}|_{s=1}\, dp \\ &\leq \iint_{P\times P} \mathrm{G}K_{(Z,Z')}|_{s=0}\, dp = \mathfrak{a}e^{\mathfrak{b}},\end{aligned} \tag{2.35}$$

this the desired (2.30).

To establish (2.34), it proves convenient to introduce the operators $\partial_a^+ \equiv (\partial_a^1 + \partial_a^2)$ and $\partial_a^- = (\partial_a^1 - \partial_a^2)$. The operator A can be written using the latter as

$$A = a_+ \partial_a^+ \partial_a^+ + a_- \partial_a^- \partial_a^- + a_{+-}(\partial_a^+ \partial_a^- + \partial_a^- \partial_a^+), \tag{2.36}$$

where

$$\bullet \quad a_+ = \frac{1}{4}(a(z,z) + a(z',z') + 2a(z,z')). \tag{2.37}$$

$$\bullet \quad a_- = \frac{1}{4}(a(z,z) + a(z',z') - 2a(z,z')).$$

$$\bullet \quad a_{+-} = \frac{1}{4}(a(z,z) - a(z',z')).$$

Note for use in what follows that both $|a_-|$ and $|a_{+-}|$ are bounded by

$$\frac{1}{4}|a(z,z) - a(z,z')| + \frac{1}{4}|a(z',z') - a(z,z')|. \tag{2.38}$$

The first point to make is that the function $\mathrm{dist}(p,p')$ is Lipschitz on some fixed radius, tubular neighborhood of the diagonal in $P \times P$ and smooth in this tubular neighborhood on the complement of the diagonal. Let U denote this tubular neighborhood. Granted this, note that $\partial_a^+ \mathrm{dist}(p,p')$ is zero on the diagonal. Thus, $\partial_a^+(\mathrm{dist}(p,p')^2)$ can be written as $u_a(p,p') \cdot \mathrm{dist}(p,p')^2$, where u_a is a Lipschitz function on U. As a consequence,

$$|\partial_a^+ \partial_a^+ \mathrm{dist}(p,p')^2| \le c_1 \mathrm{dist}(p,p')^2, \tag{2.39}$$

where c_1 is independent of p and p'. Equations (2.39) and (2.38) imply that

$$|A\mathrm{G}| \le c_2(\mathfrak{a}^{1/n} F^{n-1} + F^n) e^{\mathfrak{b}(1-s)}, \tag{2.40}$$

where c_2 is also independent of p and p'. With Holder's inequality, the preceding finds

$$|A\mathrm{G}| \le \mathfrak{a} e^{\mathfrak{b}(1-s)} + (1 + c_2^{n/(n-1)}) \cdot F^n e^{\mathfrak{b}(1-s)}. \tag{2.41}$$

Meanwhile,

$$\partial_s \mathrm{G} = -\mathfrak{a} e^{\mathfrak{b}(1-s)} - \mathfrak{b}\mathrm{G} \le -\mathfrak{a} e^{\mathfrak{b}(1-s)} - \mathfrak{b} F^n e^{\mathfrak{b}(1-s)}. \tag{2.42}$$

Thus, (2.34) follows if $\mathfrak{b}$ is chosen to be greater than the combination $(1 + c_2^{n/(n-1)})$ that appear in (2.41). q.e.d.

3. Some properties of the Gaussian measure

It proves convenient in what follows to introduce the following notation: Given a positive integer N, a point $z = (z_1, \ldots, z_N) \in \times_N M$ with pairwise distinct entries, and a bounded function F on $\times_N P$, set

$$\langle F_z \rangle_* \equiv \int_P \int_{\times_N R^d} ((\Pi_{1\le j\le N} T^{u^j})^* F)(q, \ldots, q) \cdot \exp\left[-\frac{1}{4}\mathfrak{a}_z^{-1}(u,u)\right] (4\pi)^{-N/2} \det(\mathfrak{a}_z)^{-1/2} \Pi_j du^j dq. \tag{3.1}$$

with $\mathfrak{a}_z^{-1}(u,u)$ short hand for $\sum_{1\le i\le j}(\mathfrak{a}_z^{-1})_{i\cdot j} \sum_a u_a^i u_a^j$. Note that (3.1) and (1.8) are identical when $F = \times_{1\le j\le N} \chi_{U_j}$.

Granted this notation, there are two things that must be proved so that Kolmogorov's construction can be employed to obtain a probability measure on $\times_N P$ from the various versions of (1.8): The integral in (3.1) must be non-negative when $F \ge 0$, and

$$\langle F_{(z_1,\ldots,z_N)} \rangle_* = \langle F_{(z_1,\ldots,z_{N-1})} \rangle_* \tag{3.2}$$

when F has no dependence on the point on the N'th factor in $\times_N P$. This understood, the positivity condition follows from the evident positivity of the integrand in (1.8). Meanwhile, the condition in (3.2) follows using standard properties of Gaussian integrals on $\mathbb{R}^N$.

Use $\mathcal{P}_*$ to denote the probability measure on P^M as just defined. Integrals with respect to this measure are denoted by $\langle \cdot \rangle_*$.

The remainder of this section considers the continuity of the maps that lie in the support of $\mathcal{P}_*$. The results are summarized by the lemma and theorem that follow. Neither should surprise those familiar Gaussian integrals.

Lemma 3.1. *Fix a positive integer N, a continuous function F on $\times_N P$, and positive numbers ε and δ. There exists some neighborhood of the diagonal embedding of $\times_N M$ in $(\times_N M) \times (\times_N M)$ whose points have the following property: If (z, w) is in this neighborhood, then the set of maps in P^M where $|F_z(\cdot) - F_w(\cdot)| > \delta$ has measure less than ε.*

Theorem 3.2. *The measure $\mathcal{P}_*$ induces a measure on the space of continuous maps from M to P if M is a smooth, compact manifold and the function a that appears in (1.1) is Holder continuous for some positive exponent.*

Proof of Lemma 3.1. The proof is essentially identical to the proof above of Lemma 2.3 granted the following assertion: Fix a positive integer N and a continuous function F on $\times_N P$. Then, the assignment of $z \in \times_N M$ to $\langle F_z \rangle_*$ gives a continuous function on $\times_N M$.

To see this sort of continuity, remark first that the continuity of the function a implies that of the matrix $\mathfrak{a}$ in (3.1). This implies that the assignment $z \to \mathfrak{a}^{-1}$ is continuous away from all diagonals in $\times_N M$. Thus, the assignment $z \to \langle F_z \rangle_*$ is also continuous on the complement of all diagonals in $\times_N M$.

Some notation is required so as to discuss the continuity across the diagonal. To introduce this notation, fix a non-negative integer $N' \leq N-1$. Now, let $\imath_P : \times_{N'} P \to \times_N P$ denote the embedding that has each of the first N' components of the image point equal to the corresponding component of the domain point and has remaining components equal to the final component of the domain point. Let $\imath_M : \times_{N'} M \to \times_N M$ denote the analogous embedding.

Continuity across the diagonals is implied by the following: If $z' \in \times_{N'} M$ has pairwise distinct entries, then

$$\lim_{z \to \imath_M(z')} \langle F_z \rangle_* = \langle (\imath_P * F)_{z'} \rangle_* \tag{3.3}$$

where the limit involves any sequence of points in $\times_N M$ with distinct entries that converge to $\imath(z')$.

To establish (3.3), note first that the matrix $\mathfrak{a}$ at the point $\imath(z')$ has the following block diagonal form with respect to the decomposition $\mathbb{R}^N = \mathbb{R}^{N'-1} \times \mathbb{R}^{N-N'+1}$:

$$\mathfrak{a}_{\imath(z')} = \begin{pmatrix} a' & v \\ v^T & b\Theta \end{pmatrix} \tag{3.4}$$

where the notation is as follows: First, the i, j entry of the matrix $\mathfrak{a}'$ is $a(z'_i, z'_j)$. Meanwhile, $b \equiv a(z'_{N'}, z'_{N'})$ and Θ is the matrix with every entry equal to 1. Finally, the matrix v has i, j entry given by $a(z'_i, z'_{N'})$, thus it is independent of j. As a consequence, the kernel of the matrix $\mathfrak{a}_{(\cdot)}$ at $\imath(z')$ consists of the $N' - N$ dimensional space of vectors whose first $N'-1$ entries are zero and whose final $N-N'+1$ entries sum to zero.

Since the function a is continuous, the matrix $\mathfrak{a}_z$ at a point $z \in \times_N M$ near $\imath_M(z')$ has $N - N'$ very small eigenvalues with the corresponding eigenspaces spanning a space that is very close to the kernel of the matrix that appears in (3.4). Granted all of this, standard perturbation theory for $N \times N$ matrices can be used to prove the following: If z is close to $\imath_M(z')$, then most of the mass of the integrand in (3.1) is very near the diagonal where each $k > N'$ version of u^k is equal to $u^{N'}$.

To make this last statement quantitative, let $\Pi : \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projection onto the kernel of the matrix in (3.4) and let $\Pi^\perp$ denote the orthogonal projection. What follows is the quantitative statement: There exists $\gamma > 0$ and, given $\delta > 0$, a neighborhood of

$\imath_M(z')$ whose points with distinct entries are such that

$$\sum_{i,j}(\mathfrak{a}_z^{-1})_{ij}\sum_a u_a^i u_a^j \geq \delta^{-1}\sum_a |\Pi u_a|^2 + \gamma\sum_a |\Pi^{\perp} u_a|^2. \tag{3.5}$$

Continuity as in (3.3) follows directly from this with the observation that the function on $\times_N\mathbb{R}^d$ that sends $u = (u_a^1, \ldots, u_a^N)$ to the integral of $(\Pi_{1\leq j\leq N}T^{u^j}) * F$ along the full diagonal in $\times_N P$ is continuous when F is. q.e.d.

The remainder of this section contains the

Proof of Theorem 3.2. The argument that proves Theorem 2.4 proves Theorem 3.2 granted the following replacement for Lemma 2.8:

Lemma 3.3. *Suppose that $n \geq 2$. There is a constant, κ, with the following significance: If z and z' are distinct points in M, then*

$$\langle|\phi(z)-\phi(z')|^{2n}\rangle_* \leq \kappa(|(a(z,z)-a(z,z')|^n + |a(z',z')-a(z,z')|^n). \tag{3.6}$$

Proof of Lemma 3.3. Let $F : P\times P \to [0,\infty)$ denote a smooth function that vanishes only on the diagonal and that equals $\mathrm{dist}_P(\cdot,\cdot)^2$ on some tubular neighborhood of the diagonal. The inequality in (3.6) follows from an inequality of the form

$$\langle(F^n)_{(z,z')}\rangle_* \leq \kappa'(|a(z,z)-a(z,z')|^n + |a(z',z')-a(z,z')^n| \tag{3.7}$$

where κ' is independent of z and z'. The task is to establish (3.7). To start, it is sufficient to establish (3.7) when (z,z') lie in some small radius tubular neighborhood of the diagonal in $M\times M$. This understood, write

$$a(z,z) = \alpha+\beta+\gamma,\ a(z',z') = \alpha-\beta+\gamma \text{ and } a(z,z') = \alpha-\gamma, \tag{3.8}$$

where α is bounded away from zero near the diagonal and both β and γ vanish on the diagonal. Note that the determinant of the 2×2 matrix $\mathfrak{a}$ is $2\alpha\gamma - \beta^2$.

To continue, suppose that $(u^1,u^2)\in\mathbb{R}^d\times\mathbb{R}^d$ and introduce the points $u^+ = u^1+u^2 \in \mathbb{R}^d$ and $u^- \equiv u^1 - u^2 \in \mathbb{R}^d$. Written in terms of $u^{\pm}$, one has

$$(\mathfrak{a}_z^{-1})_{ij}u_a^i u_a^j = \frac{1}{(2\alpha\gamma-\beta^2)}(\gamma u_a^+u_a^+ + \alpha u_a^-u_a^- + \beta u_a^+u_a^-). \tag{3.9}$$

Here and below the repeated subscripts are implicitly summed with $1\leq a\leq d$. In particular, (3.9) implies that

$$(\mathfrak{a}_z^{-1})_{ij}u_a^i u_a^j \geq \frac{1}{(2\alpha\gamma-\beta^2)}\left(\left(\gamma - \frac{\beta^2}{2\alpha}\right)u_a^+u_a^+ + \frac{1}{2}\alpha u_a^-u_a^-\right) \tag{3.10}$$

and thus

$$(\mathfrak{a}_z^{-1})_{ij} u_a^i u_a^j \geq \frac{1}{2\alpha} u_a^+ u_a^+ + \frac{1}{4} \frac{1}{(\gamma - \frac{1}{2\alpha}\beta^2)} u_a^- u_a^- . \tag{3.11}$$

Now, assuming that $|u^-|$ is small,

$$(T^{u^1} \times T^{u^2})^* F = |u^-|^2 + \mathcal{O}(|u^-|^3), \tag{3.12}$$

and in general, there exist positive constants r and R such that

$$(T^{u^1} \times T^{u^2})^* F \leq R \frac{r|u^-|^2}{1 + R|u^-|^2}. \tag{3.13}$$

Because of (3.11) and (3.13), the version of (3.1) that gives the left-hand side of (3.7) is bounded by a fixed multiple of $(2\alpha\gamma - \beta^2)^n$ when (z, z') is near the diagonal. This implies the inequality in (3.7) since

$$\begin{aligned} |2\alpha\gamma - \beta^2| &\leq \alpha(|2\gamma + \beta| + |2\gamma - \beta|) \\ &= \alpha(|a(z,z) - a(z,z')| + |a(z',z') - a(z,z')|). \end{aligned} \tag{3.14}$$

q.e.d.

4. Quantum field theory

This section concerns the measures from the construction in Section 1 and, by comparison, the Gaussian measure in the special cases when $M = \mathbb{R} \times Y$ where Y is a smooth, compact Riemannian manifold. It is also assumed here that the function a that appears in (1.1) is a positive Green's function for an operator on $C^\infty(\mathbb{R} \times Y)$ that has the form given in (1.9). The purpose of the ensuing discussion is to explain how the measure on $P^{\mathbb{R}\times Y}$ can be used to construct a Hamiltonian quantum field theory.

Saying this precisely requires the digression that follows to set the stage. To start the digression, suppose that Y and Ξ are a given pair of manifolds, and to keep things simple, assume that Ξ is compact. Introduce $\mathcal{F}^0$ to denote the set of functions on $\Xi^{\mathbb{R}\times Y}$ whose elements have the form $\phi \to F_z[\phi] \equiv f_1(\phi(z_1)) \cdots f_N(\phi(z_N))$; here N can be any non-negative integer, $F = (f_1, \ldots, f_N)$ any N-tuple of complex valued, continuous functions on Ξ, and $z = (z_1, \ldots, z_N)$ any point in $\times_N(\mathbb{R}\times Y)$. Let $\mathcal{F}$ denote the vector space of finite linear combinations of functions from $\mathcal{F}^0$. Note that $\mathcal{F}$ is an algebra with unit, the constant function. Now, define the subalgebra, $\mathcal{F}_+ \subset \mathcal{F}$ that is generated by those $F_z[\cdot]$ where each entry of z has non-negative $\mathbb{R}$ coordinate.

The abelian group $\mathbb{R}$ acts on $\times_N(\mathbb{R}\times Y)$ by simultaneously translating the $\mathbb{R}$-coordinate of each entry. The image of a given point z under the action of $\tau \in \mathbb{R}$ is denoted in what follows by $\tau \cdot z$. For example, if

$(t, y) \in \mathbb{R} \times Y$, then $\tau \cdot (t, y) = (t + \tau, y)$. This $\mathbb{R}$ action can be used to define an $\mathbb{R}$ action on $\mathcal{F}^0$, this the action whereby $\tau \in \mathbb{R}$ sends any given F_z to $F_{\tau \cdot z}$. This action extends by linearity to an action on the algebra $\mathcal{F}$. In the latter guise, the action of τ is denoted by R_τ. Note that this action induces an action of the semi-group $[0, \infty) \subset \mathbb{R}$ on the subalgebra $\mathcal{F}_+ \subset \mathcal{F}$.

With this digression now over, what follows gives the notion that is used here of a Hamiltonian quantum field theory.

Definition 4.1. A Hamiltonian quantum field theory of functions for the space of maps from Y to Ξ consists of the following:

- A Hilbert space, denoted here by $\mathcal{H}$.
- A $\mathbb{C}$-linear vector space homorphism, ρ, from the vector space $\mathcal{F}_+$ onto a dense domain in $\mathcal{H}$.
- A strongly continuous, self-adjoint, 1-parameter contraction semi-group on $\mathcal{H}$ that fixes $\rho(1)$. Moreover, if U_τ denotes its time $\tau \geq 0$ element, then

$$U_\tau \rho(\cdot) = \rho(R_\tau(\cdot)). \tag{4.1}$$

One comment is in order before continuing. The semigroup $\tau \to U_\tau$ is generated by a closed, non-positive self-adjoint operator on $\mathcal{H}$ (see, e.g. [4]). Multiply the latter operator by -1 to obtain what is known as the 'Hamiltionian'; its existence motivates the terminology used here. In this regard, note that the domain of the Hamiltonian consists of all vectors $\Psi \in \mathcal{H}$ for which $\lim_{\tau \to 0} \frac{1}{\tau}(1 - U_\tau)\Psi$ exists as a vector in $\mathcal{H}$. For example, $U_\tau \mathcal{H}$ is in the domain of the Hamiltonian if τ is positive.

In a somewhat different context, Osterwalder and Schrader [10] introduced a notion known as 'reflection positivity' that was used subsequently (see, eg [2]) to quantize certain non-linear, vector valued wave equations. A related notion is defined momentarily in the present context to construct a Hamiltonian quantum field theory from the measure obtained by the construction in Section 1 and from the Gaussian measure on $P^{\mathbb{R} \times Y}$, and also from the induced measures on $X^{\mathbb{R} \times Y}$. The definition given below for reflection positivity refers to the abstract case where a probability measure is defined on $\Xi^{\mathbb{R} \times Y}$ where Ξ is any given space.

The definition that follows of reflection positivity refers to a certain anti-linear involution, $*$, on $\mathcal{F}$ that is defined from the involution on $\times_N(\mathbb{R} \times Y)$ whose effect is to change the sign of the $\mathbb{R}$ factor of each entry. The latter involution is also denoted by $*$. Thus, in the case that $(t, y) \in \mathbb{R} \times Y$, then $*(t, y) = (-t, y)$. This involution of $\times_N(\mathbb{R} \times Y)$ is used to define the anti-linear involution, $*$, on $\mathcal{F}^0$, that sends any given F_z to the complex conjugate of the function F_{*z}. This anti-linear involution on $\mathcal{F}^0$ then extends as the desired anti-linear involution of the algebra $\mathcal{F}$.

The definition that follows uses $\langle\cdot\rangle_{\Xi^{\mathbb{R}\times Y}}$ to denote integration with respect to a given probability measure on $\Xi^{\mathbb{R}\times Y}$. All that follows assumes implicitly that the functions from $\mathcal{F}$ are measurable with respect to this measure.

Definition 4.2. A probability measure on $\Xi^{\mathbb{R}\times Y}$ is said in what follows to be $\mathbb{R}$-invariant when

- $\langle|R_\tau\Phi|^2\rangle_{\Xi^{\mathbb{R}\times Y}}$ for all $\tau \in \mathfrak{R}$ and $\Phi \in \mathcal{F}$.
- The function $\tau \to \langle\Phi R_\tau\Phi\rangle_{\Xi^{R\times Y}}$ on $[0,\infty)$ is continuous for all $\Phi \in \mathcal{F}$.

The probability measure is reflection positive if $\langle(*\Phi)\Phi\rangle_{\Xi^{R\times Y}} \geq 0$ for all $\Phi \in \mathcal{F}_+$.

Note that a given measure is $\mathbb{R}$-invariant precisely when the $\mathbb{R}$ action on $\mathcal{F}$ via $R_{(\cdot)}$ provides what is known as a strongly continuous group of isometries with respect to the L^2 inner product that is induced by $\langle\cdot\rangle_{\Xi^{R\times Y}}$.

The theorem that follows states Osterwalder and Schrader's reconstruction theorem [**10**] in the present context. The theorem introduces the bilinear form Q on $\mathcal{F}_+$ that is defined so that $Q(\Phi,\Phi') \equiv \langle(*\Phi)\Phi'\rangle_{\Xi^{R\times Y}}$. The kernel of Q is the subspace in $\mathcal{F}_+$ of functions Φ with the property that $Q(\Phi,\Phi') = 0$ for all $\Phi' \in \mathcal{F}_+$. This subspace is denoted by $\mathrm{Ker}(Q)$. Note that Q defines a positive definite, Hermitian bilinear form on the vector space $\mathcal{F}_+/\mathrm{Ker}(Q)$. Note as well that the subalgebra $\mathcal{F}_0 \subset \mathcal{F}_+$ that is generated by functions F_z where z has the form $((o,y_1),\ldots,(o,y_N))$, acts on $\mathcal{F}_+$ so as to preserve $\mathrm{Ker}\,(Q)$.

Theorem 4.3. *Suppose that Y and Ξ are spaces, and that a given probability measure on $\Xi^{\mathbb{R}\times Y}$ is $\mathbb{R}$-invariant and reflection positive. Let $\mathcal{H}$ denote the Hilbert space completion of the vector space $\mathcal{F}_+/\mathrm{Ker}(Q)$ with respect to the inner product that is induced by Q. Then, the following is true:*

- *Multiplication of functions in $\mathcal{F}_+$ by functions from $\mathcal{F}_0$ induces an algebra homomorphism, $\hat{\rho}$, from $\mathcal{F}_0$ into the space of bounded operators on $\mathcal{H}$.*
- *The semigroup action via $R_{(\cdot)}$ of $[0,\infty)$ on $\mathcal{F}_+$ descends to an action on $\mathcal{F}_+/\mathrm{Ker}(Q)$; and the latter extends to the whole of $\mathcal{H}$ as a self-adjoint, strongly continuous contraction semigroup.*

In particular, the data consisting of $\mathcal{H}$, the quotient map, $\rho : \mathcal{F}_+ \to \mathcal{F}_+/\mathrm{Ker}(Q)$, and the aforementioned contraction semi-group define a Hamiltonian quantum field theory.

A proof is offered at the end of this section.

As explained next, the construction in Section 1 and the Gaussian measure give examples of $\mathbb{R}$-invariant, reflection positive probability measures on both $P^{\mathbb{R}\times Y}$ and $X^{\mathbb{R}\times Y}$ in the case that (1.1)'s function a is the Green's function for an operator of the form given in (1.9). In this regard, it is assumed in what follows that Y is a compact, Riemannian manifold. In addition, $\mathcal{L}$ is assumed to be a negative definite, self-adjoint operator on $L^2(Y)$ whose degree greater than twice the dimension of Y.

To be even more specific about the function a, introduce an orthonormal basis, $\{\eta_\alpha\}$, of eigenfunctions of $\mathcal{L}$; here $\mathcal{L}\eta_\alpha = -E_\alpha^2\eta_\alpha$ with $E_\alpha > 0$ in all cases. With a point $z \in \mathbb{R}\times Y$ written as $z = (t, y)$, this notation finds

$$a((t,y),(t',y')) = \sum_\alpha \frac{1}{2E_\alpha} e^{-E_\alpha|t-t'|}\eta_\alpha(y)\eta_\alpha(y'). \tag{4.2}$$

It is assumed in what follows that a is defined on the diagonal in $(\mathbb{R}\times Y)\times(\mathbb{R}\times Y)$; thus

$$\sum_\alpha{}' \frac{1}{2E_\alpha}|\eta_\alpha(y)|^2 \tag{4.3}$$

is finite for all $y \in Y$. The function a is said to be a 'regular Green's function' if it is described by (4.2) and is continuous across the diagonal in $(\mathbb{R}\times Y)\times(\mathbb{R}\times Y)$.

For example, in the case that $Y = S^1$, take $\mathcal{L}$ so that its eigenfunctions are the exponentials, $\{e^{ik\theta}\}_{k=0,\pm1,\pm2,\ldots}$ with $E_k = (k^2+m^2)^{(1+c)/2}$ where m and c are positive and independent of the index k. The corresponding version of the function a in this case is Holder continuous with some positive exponent; as a consequence, the resulting probability measure on $P^{\mathbb{R}\times S^2}$ is supported on the continuous functions from $\mathbb{R}\times S^1$ to P.

By the way, this last claim does not follow directly from Theorems 2.4 and 3.2 because M is not compact. Even so, with a compact subset of $\mathbb{R}\times S^1$ given, the arguments for these theorems can be employed with some slight cosmetic changes to prove that the measure has support on the subset of maps in $P^{\mathbb{R}\times S^1}$ whose restriction to the given set is continuous. This last fact is enough to prove the claim by virtue of the fact that the collection $\{[-L, L]\times S^1\}_{L=1,2,\ldots}$ gives an exhaustion of $\mathbb{R}\times S^1$ by compact sets.

Theorem 4.4. *The version of the construction from Section* 1 *as described above result in an* $\mathbb{R}$*-invariant, reflection positive probability measures on both* $P^{\mathbb{R}\times Y}$ *and* $X^{\mathbb{R}\times Y}$ *in the case that* Y *is a compact, Riemannian manifold, and that the function* a *is a regular Green's function. Thus, each such measure defines via Theorem* 4.3 *a Hamiltonian quantum field theory. The identical conclusion also hold for the Gaussian measure.*

Unfortunately, no second order differential operator on S^1 has a regular Green's function since the resulting sum in (4.4) does not converge. Even so, a version of what physicists might call 'renormalization' can be used to define a quantum field theory in the latter case if X and P are carefully chosen. This renormalization business is discussed in a sequel to this article.

The Hamiltonian for the cases given by Theorem 4.4 can be described in a somewhat indirect manner by giving its associated quadratic form. This task is left to the reader with the remarks that follow as assistance. First, a self-adjoint operator on a Hilbert space defines a quadratic form as follows: If H denotes the operator in question, then the quadratic form is the polarization of $\langle\Psi, H\Psi\rangle_{\mathcal{H}}$. Here, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ denotes the Hilbert space inner product. Note for reference in what follows that a dense subspace of the Hilbert space where $\langle\cdot, H(\cdot)\rangle_{\mathcal{H}}$ is finite constitutes a quadratic form domain for the operator H.

Let $\mathcal{F}_{++} \subset \mathcal{F}_{+}$ denote the subalgebra generated by functions of the form F_z where each component of z is of the form (t, y) with $t > 0$. As it turns out, the image of $\mathcal{F}_{++}$ in Theorem 4.4's Hilbert space $\mathcal{H}$ is a quadratic form domain for the Hamiltonian, but the image of $\mathcal{F}_0$ is not part of any quadratic form domain. The upcoming Proposition 4.5 describes a dense subset in the closure of the image in $\mathcal{H}$ of $\mathcal{F}_0$ on which the Hamiltonian is defined as a quadratic form. To set the stage, fix a positive integer, N, and then an N-tuple of smooth functions, $(f_1, \ldots, f_N)$, on either $Y \times P$ or $Y \times X$ as the case may be. This N-tuple defines a function on a certain domain in P^M, this the function that sends a given map ϕ to

$$\int_{\times_N Y} f_1(y_1, \phi(y_1)) \cdots f_N(y_N, \phi(y_N))dy. \tag{4.4}$$

Here, and in what follows, Y is identified implicitly with $t = 0$ slice of $\mathbb{R} \times Y$. Also, dy is shorthand for the product volume form on $\times_N Y$.

The function depicted by (4.4) is not in $\mathcal{F}$, but as explained momentarily, it does define an element in any L^k completion of $\mathcal{F}$ as defined by either of the measures under consideration for the case that $k \in [1, \infty)$. Granted this last point, it then follows that the expression in (4.4) defines a vector in the quantum Hilbert space that is obtained by completing $\mathcal{F}_+/\mathrm{Ker}(Q)$ with the quadratic form Q.

A function of the sort given in (4.4) is defined as a measurable function with the help of a discrete, Riemann sum approximation to the multiple integral. This is to say that (4.4) is viewed as a limit of elements in $\mathcal{F}$ of the form

$$\sum_{\Delta} F_{y_\Delta} \mathrm{vol}(\Delta) \tag{4.5}$$

where the notation is as follows: First, F_y for a given point $y = (y_1, \ldots, y_N)$ denotes the the function $f_1(y_1, \cdot) \cdots f_N(y_N, \cdot)$ on either $\times_N P$ or $\times_N X$. Second, the sum in (4.5) is indexed by the simplices in a small diameter triangulation of $\times_N Y$. Here, each y_Δ is a point in its labeling simplex and $\mathrm{vol}(\Delta)$ is the volume of the simplex. In this regard, a particular simplicial decomposition is fixed and then successively subdivided to provide a countable sequence of sums as in (4.5) where the maximal simplex diameter limits to zero along the sequence. Such a sequence is used to define the expression in (4.4). The existence of the limit as an integrable function is guaranteed by Lemmas 2.3 and 3.1. These lemmae are also used to prove that the limit is independent of the choice for the points $\{y_\Delta\}$, of the choice of the starting simplicial decomposition of Y, and of the precise definition of algorithm that defines its successive subdivisions.

Note that all such functions are square integrable using either the measure from the construction in Section 1 or the Gaussian measure. This follows because the set of functions that have the form in (4.4) is closed under products. Note that if Φ is defined by (4.4) from a given N-tuple $(f_1, \ldots, f_N)$, then

$$\langle \Phi \rangle = \int_{\times_N Y} \langle F_y \rangle dy \text{ and } \langle \Phi \rangle_* = \int_{\times_N Y} \langle F_y \rangle_* dy. \tag{4.6}$$

With the stage now set, consider:

Proposition 4.5. *Let $\mathcal{H}$ denote the quantum Hilbert space for any of the cases that arise in Theorem 4.3; and let $\mathcal{D}_H \subset \mathcal{H}$ denote the subspace of finite linear combinations of vectors that have the form given in (4.5). Then $\mathcal{D}_H$ is part of a quadratic form domain for the corresponding Hamiltonian.*

Theorem 4.3 is proved momentarily, Theorem 4.4 is proved in the next section. The proof of Proposition 4.5 is left to the reader.

Proof of Theorem 4.3. Only the second of the asserted points does not simply restate the definition of $\mathcal{H}$. To see how the second point comes about, note first that when Φ and Φ' are from $\mathcal{F}_+$ and $\tau \in [0,\infty)$, then $Q(R_\tau\Phi,\Phi') = Q(\Phi, R_\tau\Phi')$ by virtue of the fact that $R_{(\cdot)}$ acts isometrically with respect to $\langle\cdot\rangle_{\Xi^{\mathbb{R}\times Y}}$. Indeed, the latter fact implies that

$$\langle(*R_\tau\Phi)\Phi'\rangle_{\Xi^{\mathbb{R}\times Y}} = \langle(R_{-\tau} * \Phi')\Phi'\rangle_{\Xi^{\mathbb{R}\times Y}} = \langle(*\Phi)R_\tau\Phi'\rangle_{\Xi^{\mathbb{R}\times Y}}. \tag{4.7}$$

To see that $R(\cdot)$ acts as a contraction on the domain $\mathcal{F}_+/\mathrm{Ker}(Q)$ in $\mathcal{H}$, first use Holder's inequality to deduce the following:

$$Q(R_\tau\Phi, R_\tau\Phi) = \langle(*R_\tau\Phi)\rangle_{\Xi^{\mathbb{R}\times Y}} \leq \langle|\Phi|^2\rangle_{\Xi^{\mathbb{R}\times Y}}. \tag{4.8}$$

Thus, $R_\tau\Phi$ has a τ-independent bound for its $\mathcal{H}$-norm. This understood, write

$$Q(R_\tau\Phi, R_\tau\Phi) = Q(\Phi, R_{2\tau}\Phi) \leq Q(\Phi,\Phi)^{1/2} Q(R_{2\tau}\Phi, R_{2\tau}\Phi)^{1/2}. \tag{4.9}$$

Now, iterate this inequality some n times to find that

$$Q(R_\tau\Phi, R_\tau\Phi) \leq Q(\Phi,\Phi)^{1-2^{-n}} Q(R_{2^n\tau}\Phi, R_{2^n\tau}\Phi)^{2^{-n}}. \tag{4.10}$$

Now use the $2^n\tau$ version of (4.8) to see that

$$Q(R_\tau\Phi, R_\tau\Phi) \leq Q(\Phi,\Phi)^{1-2^{-n}} \langle|\Phi^2|\rangle_{\Xi^{\mathbb{R}\times Y}}^{2^{-n}} \tag{4.11}$$

and take the limit on the right-hand side as $n \to \infty$.

The action is strongly continuous on the domain $\mathcal{F}_+/\mathrm{Ker}(Q)$ if

$$Q(R_\tau\Phi - \Phi, R_\tau\Phi - \Phi) \tag{4.12}$$

converges to 0 as $\tau \to 0$ for all $\Phi \in \mathcal{F}_+$. Granted (4.8), this convergence follows since the function on $[0,\infty)$ that sends τ to $\langle(R_\tau\Phi)\Phi\rangle_{\Xi^{\mathbb{R}\times Y}}$ is assumed to converge to $\langle|\Phi|^2\rangle_{\Xi^{\mathbb{R}\times Y}}$ as τ converges to zero.

The extension of $\{R_\tau\}_{\tau\geq 0}$ to the whole of $\mathcal{H}$ as a 1-parameter, self-adjoint, contraction semi-group now follows from the preceding conclusions by virtue of the fact that $\mathcal{F}_+/\mathrm{Ker}(Q)$ is dense in $\mathcal{H}$. q.e.d.

5. Reflection positivity

This section contains the proof of Theorem 4.4.

5.1. Proof of Theorem 4.4 for the measure defined by $\langle\cdot\rangle$. The $\mathbb{R}$-invariance for the measure on P^M follows from the fact that the function a is as depicted in (4.2) is unchanged by the simultaneous and equal translations of the $\mathbb{R}$ coordinates, t and t', of its two entries. To elaborate, the latter sort of invariance has the following consequence: When $\tau \in \mathbb{R}, N$ and N' are positive integers, and $z \in \times_N(\mathbb{R}\times Y)$ and $z' \in \times_{N'}(\mathbb{R}\times Y)$, then $K_{(\tau\cdot z,z')} = K_{(z,(-\tau)\cdot z')}$. (Here, and in what follows, a pair such as (z,z') is viewed as a point in $\times_{N+N'}(\mathbb{R}\times Y)$.) The first point in Definition 4.2 is a consequence of this fact about K.

The remainder of the proof concerns the reflection positivity claim. This is proved in six steps.

Step 1: To start, remark that it is sufficient to establish the reflection positivity condition solely for the linear combinations of functions from $\mathcal{F}^0$ that are all defined using the same integer N. Indeed, this is a consequence of (1.6).

With the preceding understood, let Θ denote in what follows a finite set of distinct pairs of the form (z, F) where F is a smooth, C-valued function on $\times_N P$ and $z \in \times_N(\mathbb{R} \times Y)$. Here, $N \equiv N(\Theta)$ is some positive integer that depends on Θ. The measure from the construction in Section 1 is reflection positive if and only if

$$\sum_{(z,F)(z',F')\in\Theta} \int_{(\times_N P)\times(\times_N P)} \bar{F}(p)F'(p')K_{(*z,z')}(p,p')|_{s=1}dp\,dp' \geq 0 \tag{5.1}$$

for all such sets Θ. Here, the notation is such as to implicitly identify $K_{(*z,z')}$ as a generalized function on $[0,\infty)\times(\times_N P)\times(\times_N P)$ by writing its argument from $\times_{2N}P$ as $(p_1,\ldots,p_N,p'_1,\ldots,p'_N)$. Also, the symbol dp in (5.1) indicates the volume N-form on the first factor of $\times_N P$ in $\times_{2N}P$, while dp' indicates the analogous volume form on the second factor.

The steps that follow define a sequence of approximations to $K(*z,z')$ for use on the right-hand side of (5.1). In particular, the final approximation supplies an approximation to the right-hand side of (5.1) that is evidently positive by virtue of being a sum of integrals of squares.

Step 2: Fix a Riemannian metric on P with volume form ω for which the vector fields $\{\partial_a\}$ are pointwise orthonormal. Let Δ denote the corresponding Laplacian; here, the convention has Δ being non-positive. When n is a positive integer and $j \in \{1,\ldots,n\}$, use Δ^j to denote the operator on $C^\infty(\times_n P)$ that acts as Δ on the j'th entry of any given function.

Given a positive integer n, a point, $z \in \times_n(\mathbb{R}\times Y)$ and some $\varepsilon > 0$, introduce the operator

$$A_z^\varepsilon = \sum_{1\leq i,j\leq n} a(z_i,z_j)\partial_a^i\partial_a^j + \varepsilon\sum_{1\leq i\leq n}\Delta^i. \tag{5.2}$$

Note that this operator is elliptic. Thus, there exists a smooth heat kernel for A_z^ε, this the function W_z^ε on $(0,\infty)\times((\times_n P)\times(\times_n P))$ that obeys the equation

$$\begin{aligned}\frac{d}{ds}W_z^\varepsilon = & A_z^\varepsilon W_z^\varepsilon \quad \text{with} \quad W_z^\varepsilon(p_1,\ldots,p_n;q_1,\ldots,q_n)|_{s=0} \\ & = \delta(p_1,q_1)\cdots\delta(p_n,q_n).\end{aligned} \tag{5.3}$$

Given now a pair $(z, z') \in (\times_N(\mathbb{R} \times Y)) \times (\times_N(\mathbb{R} \times Y))$, introduce $K^{\varepsilon}_{(z,z')}$ to denote the solution to the heat equation

$$\frac{d}{ds} K^{\varepsilon}_{(z,z')} = A^{\varepsilon}_{(z,z')} K^{\varepsilon}_{(z,z')} \text{ with } K^{\varepsilon}_{(z,z')}|_{s=0} = \delta^{2N} \tag{5.4}$$

on $\times_{2N} P = (\times_N P) \times (\times_N P)$. Let $\imath : P \to \times_N P$ denote the embedding as the full diagonal. Then

$$K^{\varepsilon}_{(z,z')}(p, p') = \int_P W^{\varepsilon}_{(z,z')}(p, p'; \imath(\hat{q}), \imath(\hat{q})) d\hat{q}. \tag{5.5}$$

Argue now as in Theorem 3.2.6 of [12] that the assignment of $\varepsilon \in [0, \infty)$ to

$$\int_{(\times_N P) \times (\times_N P)} G(p, p') K^{\varepsilon}_{(z,z')}(p, p') dp dp' \tag{5.6}$$

defines a continuous function on $[0, \infty)$ given any fixed, continuous function G on $\times_{2N} P$. Granted that such is the case, then the condition in (5.1) for a given Θ follows if

$$\lim_{\varepsilon \to 0} \sum_{(z,F),(z',F') \in \Theta} \int_{\times_{2N} P} \bar{F}(p_1, \dots, p_N) F'(p'_1, \dots, p'_N) \cdot K^{\varepsilon}_{(*z,z')}(p_1, \dots, p_N, p'_1, \dots, p'_N)|_{s=1} \, dp \, dp' \geq 0. \tag{5.7}$$

Step 3: This step introduces a certain 1-parameter family of perturbations of the operator $A^{\varepsilon}_{(z,z')}$. To start the story, let $H^s(p, q)$ denote the time $s \geq 0$ heat kernel for the Laplacian on P. This heat kernel is used here to approximate the operators $\{\partial_a\}$ by operators that are bounded and smoothing on $L^2(P)$. For this purpose, fix $\delta > 0$ and define the operator $\partial_{\delta,a}$ by the following rule: When $f \in C^{\infty}(P)$, then

$$(\partial_{\delta,a} f)(p) = \int_{P \times P} H^{\delta}(p, q) \partial_a H^{\delta}(q, q') f(q') dq dq'. \tag{5.8}$$

Note that as defined, $\partial_{\delta,a}$ is antisymmetric with respect to the L^2 inner product on P.

To define the desired perturbations of $A^{\varepsilon}_{(z,z')}$, remark that the latter can be written in terms of the operators A^{ε}_z and $A^{\varepsilon}_{z'}$ as

$$A^{\varepsilon}_z + A'^{\varepsilon}_{z'} + \sum_{1 \leq i,j \leq N} a(z_i, {z'}_j) \partial^i_a {\partial'}^j_a, \tag{5.9}$$

where the notation is as follows: First, A^{ε}_z and ∂^j_a differentiate with respect to the coordinates from the left factor of $\times_N P$ in $(\times_N P) \times (\times_N P)$. Meanwhile $A'^{\varepsilon}_{z'}$ and ∂'^j_a only differentiate with respect to the coordinates from the right $\times_N P$.

Granted this notation, fix $\delta \in [0,1)$ and, given points z and z' in $\times_N P$, introduce the operators

$$\text{(5.10)} \qquad \begin{aligned} &\bullet \quad A_z^{\varepsilon,\delta} \equiv \sum_{1\le i,j\le N} a(z_i,z_j)\partial^i_{\delta a}\partial^j_{\delta a} + \varepsilon \sum_{1\le i\le N} \Delta^i \\ &\bullet \quad A_{(z,z')}^{\varepsilon,\delta} \equiv A_z^{\varepsilon,\delta} + {A'}_{z'}^{\varepsilon,\delta} + \sum_{1\le i,j\le N} a(z_i,z'_j)\partial^i_{\delta a}{\partial'}^j_{\delta a}. \end{aligned}$$

This operator is elliptic and negative semi-definite for any fixed δ. As a consequence, there exists a smooth heat kernel, $W^{\varepsilon,\delta}_{(z,z')}$ on $(0,\infty)\times((\times_N P)\times(\times_N P))$ to the heat equation that is defined by $A^{\varepsilon,\delta}_{(z,z')}$.

Of prime import is the following lemma:

Lemma 5.1. *Fix $\varepsilon > 0$ and a continuous function, G, on $(\times_N P)\times(\times_N P)$. Then, the assignment of $\delta \in [0,1)$ and $(q,q') \in (\times_N P)\times(\times_N P)$ to*

$$\text{(5.11)} \qquad \int_{(\times_N P)\times(\times_N P)} G(p,p')W^{\varepsilon,\delta}_{(z,z')}((p,p'),(q,q'))|_{s=1}dp\, dp'$$

defines a continuous function on $[0,1)\times(\times_N P)\times(\times_N P)$.

Proof of Lemma 5.1. Theorem 2.12 of Chapter IX in [**5**] asserts that the function in question is continuous when viewed as a map from $[0,1)$ to $L^2((\times_N P)\times(\times_N P))$. Standard elliptic regularity theorems can be employed to prove that the expression in (5.11) defines a bonafide continuous function on $[0,1)\times((\times_N P)\times(\times_N P))$. q.e.d.

The lemma that follows is a corollary. Note that it introduces the use of $H^s_N(\cdot,\cdot)$ to denote the time s version of the heat kernel for $\sum_{1\le i\le N}\Delta^i$ on $\times_N P$.

Lemma 5.2. *The inequality in (5.1) holds if, given $r > 0$, there exists $\varepsilon_r > 0$ and, given $\varepsilon \in (0,\varepsilon_r]$, there exists $\delta_{r,\varepsilon} > 0$ with the following significance: If $\delta \in (0,\delta_{r,\varepsilon}]$, then*

$$\text{(5.12)} \qquad \sum_{(z,F),(z',F')\in\Theta} \int_{\times_4(\times_N P)} (\bar F(p)F'(p')W^{\varepsilon,\delta}_{(*z,z')}((p,p'),(q,q'))|_{s=1} \cdot Q^\delta(q,q')\, dp\, dp'\, dq\, dq' > -r$$

where Q^δ is the following approximation to δ^{2N}:

$$\text{(5.13)} \qquad Q^\delta(q,q') = \int_P H^\delta_N(q,\imath(\hat q))H^\delta_N(q',\imath(\hat q))d\hat q.$$

Part 4: The heat kernel $W^{\varepsilon,\delta}$ has been introduced for the purpose of bringing analytic perturbation theory to bear. To elaborate, first write the $(*z, z')$ version of (5.10)'s operator $A^{\varepsilon,\delta}_{(*z,z')}$

$$A^{\varepsilon,\delta}_{(*z,z')} = A^{\varepsilon,\delta}_{z} + A'^{\varepsilon,\delta}_{z'} + B. \tag{5.14}$$

Note that by virtue of the fact that B is bounded, there is, as indicated in Section 2.1 of Chapter IX in [**5**], a convergent expansion for the heat kernel $W^{\varepsilon,\delta}_{(*z,z')}$ that can be written schematically as

$$\begin{aligned}W^{\varepsilon,\delta}_{(*z,z')} = W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'} &+ \int_0^s ds_1 (W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'})|_{s-s_1} \Diamond B \Diamond (W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z})|_{s_1}\\ &+ \int_0^s \int_0^{s_1} ds_1\, ds_2 (W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'})|_{s-s_1} \Diamond B \Diamond (W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'})|_{s_1-s_2}\\ &\quad \Diamond B \Diamond (W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'})|_{s_2} + \cdots .\end{aligned} \tag{5.15}$$

Here, the notation is as follows: First,

$$(W^{\varepsilon,\delta}_{z} \otimes W^{\varepsilon,\delta}_{z'})((p,p'),(q,q')) \equiv W^{\varepsilon,\delta}_{z}(p,q) W^{\varepsilon\delta}_{z'}(p',q'). \tag{5.16}$$

Second, $\diamond$ indicates the composition of operators on $L^2((\times_N P)\times(\times_N P))$. Note that this convergence is in the following sense: Fix a continuous function, G, on $(\times_N P)\times(\times_N P)$ and let $G * \mathbf{W}$ denote the expression in (5.11). Meanwhile, use $G * \mathbf{W}_k$ to denote the corresponding expression wherein $W^{\varepsilon,\delta}_{(*z,z')}$ is replaced by the first $k+1$ terms on the right-hand side of (5.15). Then, $\{G * \mathbf{W}_k\}_{k=0,1,\dots}$ converges in $L^2((\times_N P)\times(\times_N P))$ to $G * \mathbf{W}$.

With the preceding understood, then Lemma 5.2 has the following corollary:

Lemma 5.3. *The inequality in* (5.10) *holds if, given* $r > 0$*, there exists* $\varepsilon_r > 0$ *and, given* $\varepsilon \in (0, \varepsilon_r]$*, there exists* $\delta_{r,\varepsilon} > 0$ *with the following significance: If* $\delta \in (0, \delta_{r,\varepsilon}]$*, and* k *is a sufficiently large integer, then*

$$\begin{aligned}\sum_{(z,F)(z'F')\in\Theta} \int_{P\times(\times_N P)} &((\bar{F}\otimes F') * \mathbf{W}_k)(q,q')\\ &\cdot H^{\delta}_N(q, \imath(\hat{q})) H^{\delta}_N(q', \imath(\hat{q}))\, dq\, dq'\, d\hat{q} > -r.\end{aligned} \tag{5.17}$$

Part 5: Consider the operator

$$B = \sum_{1\le i,j\le N} a(*z_i, z'_i) \partial^i_{\delta a} \partial'^j_{\delta a} \tag{5.18}$$

that appears in (5.15). With reference to (4.3), write

$$B = \sum_{\alpha,a} \sum_{1\le i,j\le N} (\mathfrak{p}_\alpha^{z_i} \partial^i_{\delta a})(\mathfrak{p}_\alpha^{z'\,i} \partial'^j_{\delta a}), \tag{5.19}$$

using

$$\mathfrak{p}_\alpha^{z_i} = \frac{1}{2\sqrt{E_\alpha}} e^{-E_\alpha t_i} \eta_\alpha(y_i). \tag{5.20}$$

Note that the expression in (5.19) writes B as an infinite sum. Given a positive integer, c, use B_c to denote the expression that is obtained taking only the terms in the sum for B in (5.19) that involve the c smallest values from the set $\{E_\alpha\}$. Since the sum in (4.4) is convergent, the set of operators $\{B_c\}_{c=1,2,\ldots}$ converges in the operator norm on $L^2((\times_N P)\times(\times_N P))$ to the operator B.

This allows the condition in (5.17) to be replaced by that stated in the next lemma.

Lemma 5.4. *The conditions set by Lemma 5.3 for the inequality in* (4.10) *to hold are obeyed if the following is true: For positive integers, k and c, let $\mathbf{W}_{k,c}$ denote the operator that is obtained by replacing B in (5.14) with B_c and then keeping only the first $k+1$ terms. Then*

$$\sum_{(z,F),(z',F')\in\Theta} \int_{P\times_2(\times_N P)} ((\bar F \otimes F') * \mathbf{W}_{k,c})(q,q') \cdot H_N^\delta(q,\imath(\hat q)) H_N^\delta(q',\imath(\hat q))\, dq\, dq'\, d\hat q \ge 0. \tag{5.21}$$

Part 6: Here is the key to the proof of (5.21): The operator $W_{k,c}$ is a finite sum, indexed by non-negative integers and, for each integer, a finite set; this sum has the form

$$\mathbf{W}_{k,b}((p,p'),(q,q')) = \sum_{0\le n}\sum_{C} \int_{\Delta^n} \mathbf{V}^z_{C,\vec s}(p,q)\mathbf{V}^{z'}_{C,\vec s}(p',q'). \tag{5.22}$$

Here, $\Delta^n \subset [0,1]^n$ denotes the n-simplex and $\mathbf{V}^z_{C,\vec s}$ is a smooth integral kernel that has continuous dependence on $z \in \mathbb{R}\times Y$ and $\vec s \in \Delta^n$. Indeed, this follows directly from (5.15) and (5.19). As a consequence, the expression in (5.21) can be written as

$$\sum_{0\le n}\sum_{C}\int_{\Delta^n} d^n s \left(\sum_{(z,F),(z',F')\in\Theta} \int_{\times_N P} (\bar F * \mathbf{V}^z_{C,\vec s} * H^\delta_N)(\imath(\hat q)) \cdot (F' * \mathbf{V}^{z'}_{C,\vec s} * H^\delta_N)(\imath(\hat q)) d\hat q \right). \tag{5.23}$$

This last expression is non-negative since it is a sum of integrals of squares,

$$\sum_{0\leq n}\sum_{C}\int_{\Delta^n} d^n s \left\| \imath * \left(\sum_{(z,F)\in\Theta} F * \mathbf{V}^{z}_{C,\vec{s}} * H^{\delta}_{N} \right) \right\|^2_{L^2}. \tag{5.24}$$

Proof of Theorem 4.4 for the Gaussian measure. The assertions for the Gaussian measure follow from the fact that the analogous assertions hold for the originating measure on Maps $(\mathbb{R}\times Y,\mathbb{R}^d)$. A proof for the latter measure can be found, for example, in Chapter 6 of [**2**]. q.e.d.

6. Generalizations

This section briefly describes some generalizations of the measure that is described by the construction in Section 1. For this purpose, return to the setting as given in Section 1. Thus, M is a topological space with a given function a as in (1.1). Meanwhile, X is a smooth, compact Riemannian manifold and $P\to X$ is a fiber bundle that meets the criteria set out in (1.2). The generalizations that follow require one additional input from M, this a continuous map, θ, from either M to P or M to X.

With $\theta : M\to P$ given, the analog of the construction in Section 1 uses solutions to (1.5) subject to a different initial value constraint at $s=0$. To elaborate, fix a positive integer, N, and a point $z=(z_1,\ldots,z_N)\in\times_N M$ with pairwise distinct entries. With z given, arguments much like those used in Section 2 can be used to prove the following: There is a unique, measure valued solution, K^θ_z, to (1.5) that is equal to the Dirac delta function on $\times_N P$ with support at $(\theta(z_1),\ldots,\theta(z_N))$ when $s=0$. Thus,

$$\int_{\times_N P} F K^\theta_z|_{s=0}\, dp = F(\theta(z_1),\ldots,\theta(z_N)). \tag{6.1}$$

In the case that $\theta : M\to X$ is given, use K^θ_z to denote the measure valued solution to (1.5) that is equal at time zero to the Dirac delta function on $\times_N P$ with support on the inverse image of the point $(\theta(z_1),\ldots,\theta(z_N))\in\times_N X$. To be precise here, remark first that the fiber of the projection $\pi : P\to X$ has a canonical volume element, this obtained by contracting the form dp that appears in (1.2) with $\partial_1\wedge\partial_2\wedge\cdots\wedge\partial_d$. Let df denote the product form on any given fiber of the induced projection from $\times_N P$ to $\times_N X$. Also, use $\pi^{-1}(\theta(z))$ to denote the fiber in $\times_N P$ over the point $(\theta(z_1),\ldots,\theta(z_N))$ in $\times_N X$. Granted this notation, the initial condition for K^θ_z is defined by the following rule:

$$\int_{\times_N P} F K_z^\theta|_{s=0}\, dp = \int_{\pi^{-1}(\theta(z))} F\, df. \tag{6.2}$$

The theorem that follows uses these integrals to generalize the measures that arise via the construction in Section 1.

Theorem 6.1. *In the case that θ maps M to P, there is a probability measure on P^M with the following properties: Fix a positive integer N, a point $z = (z_1, \ldots, z_N) \in \times_N M$ with pairwise distinct entries, and N open sets, $U_1, \ldots, U_N$, in P. The measure of the subset of maps that send each z_k to the corresponding U_k is*

$$\int_{\times_{1\le j\le N} U_j} K_z^\theta|_{s=1}\, dp. \tag{6.3}$$

This measure is supported on the set of continuous maps from M to P if M is a compact, Riemannian manifold and if both the function a *that appears in (1.1) and the map θ are uniformly Holder continuous. In any event, its push-forward defines a probability measure on X^M.*

In the case that θ maps M to X, there exists a probability measure on X^M with the following properties: Fix a positive integer N, a point $z = (z_1, \ldots, z_N) \in \times_N M$ with pairwise distinct entries, and N open sets, $V_1, \ldots, V_N$, in P. The measure of the subset of maps that send each z_k to the corresponding V_k is given by the version of (6.3) where each U_j is set equal to $\pi^{-1}(V_j)$. This measure is supported on the set of continuous maps from M to X if M is a compact, Riemannian manifold and if both the function a *that appears in (1.1) and the map θ are uniformly Holder continuous.*

Theorem 6.1 is proved by arguments that differ only cosmetically from those used in Section 2; for this reason, the details are left to the reader.

The set of measures on P^M that are defined by maps θ from M to P exhibit an amusing 'Markov' property. To elaborate, fix $\tau > 0$ and note that a measure on P^M can be obtained by replacing the condition $s = 1$ in (6.3) by the condition $s = \tau$. This requires no extra work as it amounts to replacing the function a that appears in (1.1) with τa. This understood, let $\langle\cdot\rangle^{\theta,\tau}$ denote the measure that is defined by θ for a given value of τ. Suppose that function a is Holder continuous so that each such measure is supported on the space of continuous maps from M to P. Now, fix a non-negative integer N and a function F on $\times_N P$. Then the map $\theta' \to \langle F\rangle^{\theta',\tau'}$ is measurable with respect to any given

(θ, τ) version of $\langle\cdot\rangle^{\theta,\tau}$ and

$$\langle\langle F\rangle^{(\cdot),\tau'}\rangle^{\theta,\tau} = \langle F\rangle^{\theta,\tau'+\tau}. \tag{6.4}$$

Indeed, given the definition of $\langle\cdot\rangle^{\theta,\tau}$, the equality in (6.4) follows from the Markov property of the heat kernel for (1.4)'s operator A.

The Gaussian measure from Section 1 has an analogous generalization. However, the latter does not generally exhibit the Markov property expressed by (6.4).

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